MATH 590: QUIZ 3

Name:

Throughout V will denote a vector space over F, where $F = \mathbb{R}$ or $F = \mathbb{C}$.

1. Determine if the vectors $v_1 := \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, v_2 := \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}, v_3 := \begin{pmatrix} 7 \\ 5 \\ 0 \end{pmatrix}$ are linearly independent or linearly dependent. If dependent, exhibit a dependence relation among them. (4 points)

Solution. The vectors are linearly independent if and only if the system $A \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ has a unique (zero) solution. Otherwise the vectors are linearly dependent and any solution to the system of equations gives rise to a dependence relation, where A is the 3×3 matrix whose columns are v_1, v_2, v_3. Applying Gaussian elimination to the augmented matrix $\left(\begin{array}{ccc|c} 1 & 3 & 7 & 0 \\ 1 & 2 & 5 & 0 \\ 2 & -1 & 0 & 0 \end{array}\right)$ produces the reduced row echelon matrix $\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$. This shows that the corresponding homogeneous system of equations has a non-zero solution, and thus the vectors v_1, v_2, v_3 are linearly dependent.

The solutions to the system are all vectors of the form $\begin{pmatrix} -t \\ -2t \\ t \end{pmatrix}$, with $t \in \mathbb{R}$. Taking $t = -1$, we have $1 \cdot v_1 + 2 \cdot v_2 - 1 \cdot v_3 = \vec{0}$.

2. State the Exchange Theorem. (3 points)

Solution. Given vectors $w_1, \ldots, w_r, u_1, \ldots, u_s \in V$ such that $\text{Span}\{w_1, \ldots, w_r\} = V$ and $u_1, \ldots, u_s$ are linearly independent, then $s \leq r$. Moreover, after re-indexing the $w_i's$, we have $V = \text{Span}\{u_1, \ldots, u_s, w_{s+1}, \ldots, w_r\}$.

3. Find a basis for the space of real symmetric 2×2 matrices. **No need to justify your answer**. (3 points)

Solution. A (natural) basis is $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. This follows since given a real symmetric matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$, we have

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

and the given matrices are clearly linearly independent.